Lecture 12. 4 juin 2025

Recall: (X, g^{TX}) oriented Riemannian manifold

$C(TX)$ vector bundle of Clifford algebra

$(E = E^+ \oplus E^-, \ h^E = h^{E^+} \oplus h^{E^-}, \ \nabla^E = \nabla^{E^+} \oplus \nabla^{E^-})$

$\mathbb{Z}_2$-graded Clifford module ∇^E Hermitian Clifford connection

self-adj: $v \in TX$ $\quad c(v)^* = -c(v)$

Def: Dirac operator

$$D^E := \sum_j c(e_j) \nabla^E_{e_j}$$

$\{e_j\}$ ONB of (TX, g^{TX})

L^2-inner product on $C^\infty_c(X, E)$

$$\langle s_1, s_2 \rangle_{L^2} = \int_X \langle s_1, s_2 \rangle_{h^E}(x) \, dV(x)$$

Riem. volume form

Prop: D^E is a symmetric elliptic operator of order 1.

$\forall s_1, s_2 \in C^\infty_c(X, E)$ $\quad \langle D^E s_1, s_2 \rangle_{L^2} = \langle s_1, D^E s_2 \rangle$

Lemma: If $\beta \in \Omega^1_c(X)$, set

$$\mathrm{Tr}[\nabla^{T^*X} \beta] := \sum_j e_j \beta(e_j) - \beta\Big(\sum_j \nabla^{TX}_{e_j} e_j\Big)$$

$\{e_j\}$ ONB of (TX, g^{TX})

Then
$$\int_X \underbrace{\mathrm{Tr}[\nabla^{T^*X} \beta]}_{\text{has cpt support}} \, dV(x) = 0$$

Pf: (X, g^{TX})

$\exists !$ $W \in C^\infty(X, TX)$ s.t.

$$\beta(U) = g^{TX}(W, U) \quad \forall U \in TX.$$

Recall $dv(x) = e^1 \wedge \cdots \wedge e^m$ $\quad m = \dim X$

$$L_W dV = \sum_j e^1 \wedge \cdots \wedge e^{j-1} \wedge L_W e^j \wedge \cdots \wedge e^m$$

$$= \sum_j \langle L_W e^j, e_j \rangle \underbrace{e^1 \wedge \cdots \wedge e^m}_{dV(x)}$$

$$\langle L_W e^j, e_j \rangle = W \underbrace{\langle e^j, e_j \rangle}_{=1} - \langle e^j, L_W e_j \rangle$$

$$= - \langle e_j, [W, e_j] \rangle_{g^{TX}}$$

$$= - \langle e_j, \nabla^{TX}_W e_j - \nabla^{TX}_{e_j} W \rangle_{g^{TX}}$$

$$= - \underbrace{\langle e_j, \nabla^{TX}_W e_j \rangle_{g^{TX}}}_{=0} + \langle e_j, \nabla^{TX}_{e_j} W \rangle_{g^{TX}}$$

$$= e_j \langle e_j, W \rangle_{g^{TX}} - \langle W, \nabla^{TX}_{e_j} e_j \rangle_{g^{TX}}$$

$$= e_j \beta(e_j) - \beta(\nabla^{TX}_{e_j} e_j)$$

$$\Rightarrow L_W dv(x) = \mathrm{Tr}[\nabla^{T^*X} \beta]\, dv(x)$$

$$\int_X \mathrm{Tr}[\nabla^{T^*X} \beta]\, dv(x) = \int_X L_W \, dv(x)$$

$$= \int_X (d\iota_W + \iota_W d)(dv(x))$$

$$= \int_X d\iota_W \, dv(x) \equiv 0 \qquad \#$$

$\iota_W dv(x)$ has cpt supp

Proof of Prop: $m = \dim_{\mathbb{R}} X$

① D^E is first order elliptic diff. op.

$(U, (x_1, \cdots, x_m))$ local chart on X s.t.
$\in \mathbb{R}^m$

$$E|_U \simeq U \times \mathbb{C}^r \qquad r = \mathrm{rk}\, E$$

$$\nabla^E = d + \Gamma^E \qquad \Gamma^E \in \Omega^1(U, \mathrm{End}(\mathbb{C}^r))$$

Put $G(x) = (g_{ij}(x))$ symmetric matrix > 0

$$g_{ij}(x) = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_{g^{TX}}(x) \qquad (g^{ij}(x)) = G^{-1}(x)$$

Put $G^{-1/2}(x) = (b^{ij}(x))$

Then $$e_i(x) = \sum_{j=1}^m b^{ij}(x) \frac{\partial}{\partial x_j}$$

$\{e_i\}$ ONB of $(TX, g^{TX})|_U$

So on U

$$\begin{aligned} D^E &= \sum_j c(e_j) \nabla^E_{e_j} \\ &= \sum c\left(b^{jk} \frac{\partial}{\partial x_k}\right)\left(b^{jl} \frac{\partial}{\partial x_l} + \Gamma^E\left(b^{jl} \frac{\partial}{\partial x_l}\right)\right) \\ &= \sum_{k,l} \underbrace{\left(\sum_j b^{jk} b^{jl}\right)}_{g^{kl}} c\left(\frac{\partial}{\partial x_k}\right)\left(\frac{\partial}{\partial x_l} + \Gamma^E\left(\frac{\partial}{\partial x_l}\right)\right) \end{aligned}$$

$$\begin{aligned} \sigma(D^E)(x, \xi) &= \sum_{k,l} g^{kl}(x)\, c\left(\frac{\partial}{\partial x_k}\right)\left(\sqrt{-1} \left\langle \xi, \frac{\partial}{\partial x_l} \right\rangle\right) \\ &= \sum_j c(e_j)(\sqrt{-1} \langle \xi, e_j \rangle) = \sqrt{-1}\, c(\xi^*) \end{aligned}$$

$\xi \in T^*_x X$

$\xi^* \in TX$ metric dual of ξ

$$\langle \xi^*, u \rangle_{g^{TX}} := (\xi, u) \qquad \forall\, u \in TX$$

Clearly $\sigma(D^E)(x,\xi)^2 = -c(\xi^*)^2$ (4)

$$= |\xi|^2 \, Id_E \curvearrowright End(E_x)$$

invertible for $\xi \neq 0$

$$\Rightarrow D^E \text{ is elliptic.}$$

(2) D^E is symmetric. For $s_1, s_2 \in C_c^\infty(X, E)$

$$\langle D^E s_1, s_2 \rangle_{h^E}(x) = \sum_j \langle c(e_i) \nabla^E_{e_i} s_1, s_2 \rangle_{h^E}(x)$$

$$= - \sum_j \langle \nabla^E_{e_i} s_1, c(e_i) s_2 \rangle_{h^E}(x)$$

$c(e_i)^* = -c(e_i)$

$$= -\sum_j e_i \langle s_1, c(e_i) s_2 \rangle_{h^E}(x)$$

∇^E preserves the metric

$$+ \sum_j \langle s_1, \nabla^E_{e_i} c(e_i) s_2 \rangle_{h^E}(x)$$

$$= -\sum_j \left(e_i \langle s_1, c(e_i) s_2 \rangle_{h^E}(x) - \langle s_1, c(\nabla^{TX}_{e_i} e_i) s_2 \rangle_{h^E}(x) \right)$$

$[\nabla^E_{e_i}, c(e_i)] = c(\nabla^{TX}_{e_i} e_i)$

$$+ \langle s_1, \underbrace{\sum_j c(e_i) \nabla^E_{e_i}}_{D^E} s_2 \rangle_{h^E}(x)$$

Define $\beta \in \Omega^1_0(X)$ by

$$\beta(W) := \langle s_1, c(W) s_2 \rangle_{h^E}(x)$$

$$\Rightarrow \langle D^E s_1, s_2 \rangle_{h^E}(x) = -\mathrm{Tr}[\nabla^{T^*X} \beta] + \langle s_1, D^E s_2 \rangle_{h^E}(x)$$

$\int \downarrow dv(x)$ $\rightsquigarrow$ D^E symmetric by the Lemma

#

<u>Cor</u>: $\forall f \in C^\infty(X)$

$$[D^E, f] = c(df^*)$$

<u>HW 4.5</u>

Theorem (Spectral Theory of D^E): Assume X cpt.

① D^E is Fredholm operator, that is $\operatorname{Ker} D^E$ & $\operatorname{Coker} D^E$ are finite dimensional

Rmk: $D^E : \operatorname{Dom}(E) = C^\infty(X,E) \subset L^2(X,E) \to L^2(X,E)$ unbounded linear operator.

② $(D^E)^2 \geq 0$, symmetric. $\exists$ a complete orthonormal basis $\{\phi_i\}_{i=1}^{+\infty}$ of $(L^2(X,E), \langle\, ,\, \rangle_{L^2})$

s.t. $(D^E)^2 \phi_i = \lambda_i \phi_i$

with $\begin{cases} 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \\ \lambda_k \to +\infty \end{cases}$

$\operatorname{Spec}((D^E)^2) = \{$ eigenvalues of finite multiplicities $\}$

③ ellipticity of $(D^E)^2 \Rightarrow$ each $\phi_i \in C^\infty(X,E)$.

KEY pt: Sobolev embedding Thm & Elliptic estimates

$((D^E)^2 + 1)^{-1}$ is a compact linear operator on $L^2(X,E)$.

V.2) Spin structure and spin manifold

G (compact) Lie group.

Def (G - Principal bundle) A principal bundle P on X with structure group G is a fiber bundle P with a right action of G on the fibers, that is

⑥

$$\begin{cases} \cdot \ X = \bigcup_\alpha V_\alpha \\ \cdot \ P|_{V_\alpha} \simeq V_\alpha \times G \\ \text{transition fct} : V_\alpha \cap V_\beta \to G \end{cases}$$

acting on fibres G on the left.

In this case $P/G \cong X$.

Examples: ① $E \to X$ complex vector bundle of $\mathrm{rk} = r$

$GL(E) \to X$ frame bundle of E

$\forall x \in X$

$GL(E)_x := \{\alpha \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, E_x) : \alpha \text{ invertible}\}$

str. gp $= GL(r, \mathbb{C})$

② Given h^E Hermitian metric on E

$\rightsquigarrow$ $U(E)$ = unitary frame bundle

$U(E)_x$ = Isometries $(\mathbb{C}^r, \langle\rangle) \xrightarrow{\sim} (E_x, h^E_x)$

str. gp = $U(r)$ $\simeq \{$ orthonormal basis of $(E_x, h^E_x)\}$

③ (TX, g^{TX}) Riemannian metric

$\rightarrow$ $O(m)$-bundle $O(X) \to X$ $m = \dim X$

$O(X)_x := \{$ orthonormal basis of $(T_x X, g^{TX}_x)\}$

If X is oriented, $SO(X) \to X$.

oriented orthonormal basis

Prop: (X, g^{TX}) oriented Riem mfd

$SO(X)_x :=$ oriented isometries from $(\mathbb{R}^m, \langle\rangle_{Eucl})$ to (T_xX, g_x^{TX})

Then $SO(X)$ is a $SO(m)$-principal bundle on X

Moreover

$$\begin{cases} TX \simeq SO(X) \times_{SO(m)} \mathbb{R}^m \\ \qquad (p, v) \sim (pg, g^{-1} \cdot v) \quad \forall\, g \in SO(m),\ p \in SO(X),\ v \in \mathbb{R}^m \\ C(TX) \simeq SO(X) \times_{SO(m)} C(\mathbb{R}^m) \end{cases}$$

$C(\mathbb{R}^m)$: Clifford alg.

HW 5.1

Def: A spin structure on oriented (X, g^{TX}) is a $Spin(m)$-principal bundle $Spin(X)$ s.t

$$TX \simeq Spin(X) \times_{Spin(m)} \mathbb{R}^m$$

where $Spin(m) \circlearrowright \mathbb{R}^m$ by $v \mapsto a v a^{-1}$.

Note: $p : Spin(m) \to SO(m)$ double covering $(m \geq 2)$

$Spin(m) \circlearrowright \mathbb{R}^m$

$=$ via p, $SO(m) \circlearrowright \mathbb{R}^m$

So

$$SO(X) = Spin(X) \times_{Spin(m)} SO(m)$$

$\Rightarrow$ $Spin(X) \to SO(X)$ double covering

Def: If $Spin(X)$ exists, we call X a Spin manifold.

<u>Thm</u>: An oriented mfd X has a spin structure if and only if $\omega_2(X) = 0 \in H^2(X, \mathbb{Z}_2)$, where $\omega_2(X)$ denotes the second Stiefel-Whitney class.

Different spin structures $= H^1(X, \mathbb{Z}_2)$.

<u>Examples</u>: $\begin{cases} \text{Spin: Riemann surface of genus, } S^n \ (n \geq 2),\ \mathbb{CP}^{2n+1},\ \text{Calabi-Yau} \\ \text{Not spin: } \mathbb{CP}^{2n} \ (n \geq 2) \end{cases}$

When X is spin, we have spinor bundle $(S(TX), h^{S(TX)})$ and <u>even-dimensional</u> $n = \text{even}$

$$Spin(n) \circlearrowright (S^{\pm}, h^{S^{\pm}}) \quad \text{unitary}$$

$$S^{TX\pm} = Spin(X) \times_{Spin(n)} S^{\pm} \qquad \mathbb{Z}_2\text{-graded Clifford module}$$

$$h^{S^{\pm}} \rightsquigarrow h^{S^{TX\pm}}$$